Bayesian Inference

Data Evaluation and Decisions

Corrections to the book published by Springer Verlag, Heidelberg 2016

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In the last paragraph of the ***preface*** the first phrase should read: The example of the binomial distribution — sketched on Fig. 5.1 — represents 300 years of research in statistics.

Contents

Chapter 1

Knowledge and Logic

As yet there are no corrections to Chap. 1.

Chapter 2

Bayes' Theorem

Following Eq. (2.9) the text should be: The absolute value $|\ldots|$ apppears because $\mathrm{d}\xi/\mathrm{d}\eta$ may be positive or negative. It must, however, have the same sign everywhere.

In the second phrase after Eq. (2.27) it should read: The posteriors (2.20) for $N = 3$ and $N = 4$ ...

After Eq. (2.29) plus two more sentences it should read: ... lower part of the figure. The eight events are indicated by the dots on the abscissa. It is rectangular ...

Chapter 3

Probable and Improbable Data

In the caption of Fig. 3.8 the second phrase should read: This curve is a version of the chi-squared distribution with 7.4 degrees of freedom; see Sect. 4.1.3

The second last phrase of Chap. 3 should read: Let us note that the distribution w becomes a chi-squared distribution with $N = 2\alpha$ degrees of freedom . . .

Chapter 4

Description of Distributions I: Real x

In the second line of Eq. (4.5) there should be the fourth power of σ.

The third line of the paragraph that begins after Eq. (4.8) should read: ... considered as the sum of sufficiently many contributions that all follow the same distribution.

The footnote on p. 48 should be modified: ... in the first edition of this book. The present definition is easier and more commonly used.

In Fig. 4.2 the abscissa should be labelled T.

The paragraph that ends after Eq. (4.46) should be completed with the phrases: This occurs because neither the first moment $\overline{t}$ nor the second moment $\overline{t^2}$ exist of the distribution (4.45). They are necessary for z to approach a Gaussian distribution.

Chapter 5

Description of Distributions II: Natural x

There is no correction to Chap. 5.

Chapter 6

Form Invariance I

Following Eq. (6.5) insert the demand: The interested reader should answer the question:

Chapter 7

Examples of Invariant Measures

There are no corrections to Chap. 7.

Chapter 8

A Linear Representation of Form Invariance

There are no corrections to Chap. 8.

Chapter 9

Going Beyond Form Invariance: The Geometric Prior

In the second line of p. 104 it should read: Its elements estimate the second derivative of $-\ln p$,

In Eq. (9.16) the sum reaches from $\nu = 1$ to $\nu = n$.

Chapter 10

Inferring the Mean or the Standard Deviation

At the end of the introductory remarks on p. 115 there should be mentioned: Section A.10 gives the solutions to the problems suggested to the reader.

Chapter 11

Form Invariance II: Natural x

There are no corrections to Chap. 11.

Chapter 12

Item Response Theory

In the References on p. 149 under number 3 it should read: ... a theory of objectivity in comparisons ... ed. by L.J.Th. van der Kamp ...

In Ref. 14 it should read: ... Published by Springer VS, Wiesbaden 2018

Chapter 13

On the Art of Fitting

The text after Eq. (13.3) should read: with N degrees of freedom; see Sect. 4.1.3. We assume that all σ_k are equal to unity. Then the scaling parameter is

The text after Eq. (13.5) should read: If the actual T is much larger ...

The text after Eq. (13.9) should read: The position of the maximum ...

The text after Eq. (13.10) should read: The interested reader may want to verify this estimate.

The paragraph beginning before Eq. (13.11) should read: We can redefine η such that η^{ML} does not depend on N. This is reached if one shifts the parameter η to

The left hand side of Eq. (13.15) should read $\tilde{\chi}^{\mathrm{sq}}_N(y|\eta')$

Eq. (13.17) should read $q(n_k|\lambda_k) = \frac{\lambda_k^{n_k}}{n_k!}\, e^{-\lambda_k}$,

The text after Eq. (13.34) should read: The expression (13.33) is a likelihood function. ...

In both lines of Eq. (13.35) the minus should be replaced by a plus sign.

The line that introduces Eq. (13.37) should read: The Fisher function of (13.36) is

The second paragraph of Sect. 13.4 should end with the statement: ... the Shannon information of a form invariant model becomes independent of the parameterisation of the model.

Chapter 14

Summary

The last lines of the first paragraph of Sect. 14.1 should read: ... needed to make the posterior independent of the prior, [1]. As long as the prior remains arbitrary, one can generate any posterior.

In the fifth line of the last paragraph of Sect. 14.1 it should read: that is, whether the observed events $x_1, \dots, x_N$ comply with the distribution $p(x_1, \dots, x_N|\xi^{\text{pre}})$.

The second paragraph of Sect. 14.2 should read: ... The test is widely used to assess the quality of a fit. The present method yields a chi-squared criterion that is simmilar to the test. However, the criterion rejects "overfitting". It rejects fits that come too close to observation. In particular, it rejects a fit that reproduces the observed $\boldsymbol{x}$ point by point. One can do so by Occam's argument ...

Following Eq. (14.1) it should read: ... a given level and its nearest neighbour when the mean level distance is unity. It is certainly ...

In the paragraph on p. 171 that begins with "It is possible that" it should read in the 8$^{\text{th}}$ line: a model $p(s|\xi)$ that interpolates ...

The last paragraph of Sect. 14.3 should read: It remains the mystery that probability distributions are ordered by a symmetry although their events do not know of each other. They happen independently. To which extent are events independent that have a common observer?

Appendix A

Problems and Solutions

In the present appendix, we give the solutions or hints to the solutions of the problems that have been posed within the main text of the book.

A.1 Knowledge and Logic

A.1.1 The Joint Distribution

The phrase after Eq. (A.1) should be continued as follows: ... not conditioned by x_2 ; it is independent of x_2 .

The words "A proof by induction yields" should be replaced by: Since any pair of the x_k is statistically independent of each other, one obtains

A.2 Bayes' Theorem

A.2.1 Bayes' Theorem under Reparameterisations

There is no correction to this subsection.

A.2.2 Transformation to the Uniform Prior

After Eq. (A.6) one should insert: When $\mu(\xi)$ was already uniform, then η becomes a linear transformation of ξ and thus $\mu_T(\eta)$ will again be a uniform prior.

A.2.3 The Iteration of Bayes' Theorem

There is no correction to this subsection.

A.2.4 The Gaussian Model for many Events

There is no correction to this subsection.

A.2.5 The Distribution of Leading Digits

After the phrase following Eq. (A.10) one should insert: The function log is the logarithm to the basis of 10.

After Eq. (A.11) one should insert: Here, the function ln is the natural logarithm.

A.3 Probable and Improbable Data

A.3.1 The Size of an Area in Parameter Space

There is no correction to A.3.1.

A.3.2 No Decision without Risk

There is no correction to A.3.2.

A.3.3 Normalisation of a Gaussian Distribution

There is no correction to A.3.3.

A.3.4 The Measure of a Scale Invariant Model

There is no correction to A.3.4.

A.3.5 A Single Decay Event

There is no correction to A.3.5.

A.3.6 Normalisation of a Posterior of the Gaussian Model

There is no correction to A.3.6.

A.3.7 The ML Estimator from a Gaussian Likelihood Function

There is no correction to A.3.7.

A.3.8 The ML Estimator from a chi-squared Model

There is no correction to A.3.8.

A.3.9 Contour Lines

There is no correction to A.3.9.

A.3.10 The Point of Maximum Likelihood

There is no correction to A.3.10.

A.4 Description of Distributions I: Real x

A.4.1 The Mean of a Gaussian Distribution

There is no correction to A.4.1.

A.4.2 On the Variance

There is no correction to A.4.2.

A.4.3 Moments of a Gaussian

There is no correction to A.4.3.

A.4.4 The Normalisation of a Multidimenional Gaussian

There is no correction to A.4.4.

A.4.5 The Moments of the Chi-Squared Distribution

There is no correction to A.4.5.

A.4.6 Moments of the Exponential Distribution

There is no correction to A.4.6.

A.5 Description of Distributions II: Natural x

A.5.1 The Second Moments of the Multinomial Distribution

There is no correction to A.5.1.

A.5.2 A Limit of the Binomial Distribution

There is no correction to A.5.2.

A.6 Form Invariance I

A.6.1 Every Element can be Considered the Origin of a Group

There is no correction to A.6.1.

A.6.2 The Domain of Definition of a Group Parameter is Important

There is no correction to A.6.2.

A.6.3 A Parameter Representation of the Hyperbola

There is no correction to A.6.3.

A.6.4 Multiplication Functions for the Symmetry Groups of the Circle and the Hyperbola

There is no correction to A.6.4.

A.6.5 The Group of Dilations

There is no correction to A.6.5.

A.6.6 The Combination of Translations and Dilations

There is no correction to A.6.6.

A.6.7 Reversing the Order of Translation and Dilation

There is no correction to A.6.7.

A.6.8 A Transformation of the Group Parameter

There is no correction to A.6.8.

A.6.9 A Group of Transformations of the Group Parameter

There is no correction to A.6.9.

A.6.10 The Model p is normalized when the Common Form w is Normalized

There is no correction to A.6.10.

A.6.11 Two Expressions Yielding the Measure μ

There is no correction to A.6.11.

A.6.12 Form invariance of the Posterior Distribution

There is no correction to A.6.12.

A.6.13 Invariance of the Shannon Information

There is no correction to A.6.13.

A.7 Examples of Invariant Measures

A.7.1 The Invariant Measure of the Group of Translation-Dilation

There are no corrections to A.7.1.

A.7.2 Groups of Finite Volume

There are no corrections to A.7.2.

A.7.3 The Inverse of a Triangular Matrix

There are no corrections to A.7.3.

A.7.4 The Invariant Measure of a Group of Triangular Matrices

There are no corrections to A.7.4.

A.8 A Linear Representation of Form Invariance

A.8.1 Transforming a Space of Square Integrable Functions

There is no correction to A.8.1.

A.8.2 An Integral Kernel

There is no correction to A.8.2.

A.9 Beyond Form Invariance: The Geometric Prior

A.9.1 Jeffreys' Rule Transforms as a Density

There is no correction to A.9.1.

A.9.2 The Fisher Matrix is Positive Definite

There is no correction to A.9.2.

A.9.3 The Measure on the Sphere

There is no correction to A.9.3.

A.9.4 Another Form of the Measure on the Sphere

There is no correction to A.9.4.

A.10 Inferring the Mean or the Standard Deviation

A.10.1 Calculation of a Fisher Matrix

There is no correction to A.10.1.

A.10.2 The Expectation Value of an ML Estimator

There is no correction to A.10.2.

A.11 Form Invariance II: Natural x

A.11.1 The Identity of two Expressions

There is no correction to A.11.1.

A.11.2 Form Invariance of the Binomial Model

There is no correction to A.11.2.

A.11.3 The Multiplication Function of a Group of Matrices

In the the first line of A.11.3 it should read: Show that the multiplication function ϕ of the group … is

A.11.4 An ML Estimator for the Binomial Model

There is no correction to A.11.4.

A.11.5 A Prior Distribution for the Poisson Model

There is no correction to A.11.5.

A.11.6 A Limiting Case of the Poisson Model

There is no correction to A.11.6.

A.12 Item Response Theory

A.12.1 Expectation Values Given by the Binomial Model

There is no correction to A.12.1.

A.13 On the Art of Fitting

A.13.1 A Maximum Likelihood Estimator

There is no correction to A.13.1.

A.13.2 Gaussian Approximation to a Chi-Squared Model

There is no correction to A.13.2.

Appendix B

Description of Distributions I: Real x

Following Eq. (B.26) the text should read: for positive real part of z. This is given in . . .

Preceding Eq. (B.33) it should read: . . . the substitution $\beta x = t^2\gamma^{-2}$ yields . . .

Appendix C

Form Invariance I

There is no correction to Chap. C.

Appendix D

Beyond Form Invariance: The Geometric Prior

There is no correction to Chap. D.

Appendix E

Inferring Mean or Standard Deviation

There is no correction to Chap. E.

Appendix F

Form Invariance II: Natural x

There is no correction to Chap. F.

Appendix G

Item Response Theory

There is no correction to Chap. G.

Appendix H

On the Art of Fitting

We have completely rewritten the present appendix because in the second edition of the present book it was too sketchy to be understandable.

H.1 The Geometric Measure on the Scale of a Chi-Squared Distribution

According to the second line of Eq. (9.18) the geometric measure on the scale of η is

$$\mu_g(\eta) = \frac{1}{2}\left[F(\eta)\right]^{1/2}, \tag{H.1}$$

if F is the Fisher function of Eq. (9.2), which means

$$F(\eta) = -\int \mathrm{d}y\, \tilde{\chi}_f^{\mathrm{sq}}(y|\eta)\, \frac{\partial^2}{\partial\eta^2} \ln \tilde{\chi}_f^{\mathrm{sq}}(y|\eta)\,, \tag{H.2}$$

and $\tilde{\chi}_f^{\mathrm{sq}}(y|\eta)$ is the model (13.7). This yields

$$\begin{aligned} F(\eta) &= -\frac{1}{\Gamma(f/2)}\int_{-\infty}^{\infty} \mathrm{d}y\, \exp\left(\frac{f}{2}[y-\eta] - e^{y-\eta}\right) \frac{\partial^2}{\partial\eta^2}\left(\frac{f}{2}[y-\eta] - e^{y-\eta}\right) \\ &= \frac{1}{\Gamma(f/2)}\int \mathrm{d}y\, \exp\left(\frac{f}{2}[y-\eta] - e^{y-\eta}\right) e^{y-\eta} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(f/2)} \int_{-\infty}^{\infty} \mathrm{d}y \, \exp\left((\frac{f}{2}+1)[y-\eta] - e^{y-\eta} \right) \\
&= \frac{\Gamma(f/2+1)}{\Gamma(f/2)} \\
&= \frac{f}{2} \, .
\end{aligned}
\qquad \text{(H.3)}
$$

For the step from the third to the fourth line of this equation, one uses the fact that $\tilde{\chi}_N^{\mathrm{sq}}$ is normalised to unity. The last line follows from Eq. (B.24) in Appendix B. By Eq. (H.1) one obtains

$$
\mu_g(\eta) = \left(\frac{f}{8}\right)^{1/2} . \qquad \text{(H.4)}
$$

H.2 Convoluting Chi-Squared Distributions

Let a set of positive numbers t_k , $k = 1, \dots, N$, be given so that each one follows a chi-squared distribution (4.34). The distribution of t_k ,

$$
q_k(t_k|\xi) = \frac{\xi^{f_k/2}}{\Gamma(f_k/2)} \, t_k^{f_k/2-1} \exp(-t_k \xi) \, , \quad 0 < t_k, \xi < \infty \, , \qquad \text{(H.5)}
$$

shall have the number f_k of degrees of freedom. The f_k are positive; they need not be integer. For $k \neq k'$ the number f_k may be different from $f_{k'}$. We show that the quantity

$$
T = \sum_{k=1}^{N} t_k \qquad \text{(H.6)}
$$

follows the chi-squared distribution with

$$
f^{\mathrm{tot}} = \sum_{k=1}^{N} f_k \qquad \text{(H.7)}
$$

degrees of freedom. In Eq. (13.23) we have made use of this theorem. It is a consequence of the convolution theorem . We explain the notion of “convolution" and state the theorem. It describes the structure of the Fourier[1] transform of a convolution. From this follows the distribution of T , see Sect. H.4. The Fourier transform is defined in Sect. H.3.

The convolution $q_1 \circ q_2(x)$ of the functions $q_1(t_1|\xi)$ and $q_2(t_2|\xi)$ is

$$\begin{aligned} q_1 \circ q_2\,(T) &= \int_0^\infty \mathrm{d}t_2\, q_1(T-t_2|\xi)q_2(t_2|\xi) \\ &= \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2\, \delta(T-t_1-t_2)q_1(t_1|\xi)q_2(t_2|\xi) \\ 0 &< T < \infty\,, \end{aligned} \tag{H.8}$$

where $\delta(x)$ is Dirac's δ distribution. Integrating $q_1 \circ q_2$ over T from 0 to ∞ yields unity since the distributions q_k are normalised to unity. The N-fold convolution is

$$q_1 \circ \ldots \circ q_N\,(T) = \int_0^\infty \mathrm{d}t_1 \ldots \int_0^\infty \mathrm{d}t_N\, \delta(T - \sum_{k=1}^N t_k) \prod_{k=1}^N q_k(t_k)\,. \tag{H.9}$$

This convolution is again normalised to unity.

According to Eq.(H.17) the Fourier transform of q_k is

$$F_k(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \mathrm{d}t_k\, q_k(t_k|\xi)e^{izt_k}\,. \tag{H.10}$$

Here, we set $q_k(t_k|\xi)$ equal to zero for negative values of t_k in order to obtain the integration from $-\infty$ to ∞ required by the definition (H.17) of the Fourier transform. Equation (H.32) says that

$$F_k(z) = \frac{\xi^{f_k/2}}{\sqrt{2\pi}} \frac{1}{(\xi - iz)^{f_k/2}}\,. \tag{H.11}$$

Let $F_{1\circ 2}$ be the Fourier transform of $q_1 \circ q_2$. The convolution theorem says that $F_{1\circ 2}(z)$ equals the product of F_1 and F_2 ,

$$F_{1\circ 2}(z) \;=\; F_1(z)F_2(z)\,, \tag{H.12}$$

[1] Joseph Fourier, 1768–1830, French mathematician and physicist, member of the Académie des Sciences. He studied the transport of heat in solids. In this context he discovered the possibility to expand distributions into the series which nowadays carries his name.

see entry 3 in Sect. 12.22 of [2]. It follows

$$F_{1\circ 2}(z) = \frac{\xi^{(f_1+f_2)/2}}{2\pi} \frac{1}{(\xi - iz)^{(f_1+f_2)/2}} \,. \tag{H.13}$$

This can be generalised to the statement: the Fourier transform $F_{1\circ\ldots\circ N}(\xi)$ of the N-fold convolution (H.9) equals the product of the N Fourier transforms F_k ,

$$F_{1\circ\ldots\circ N}(\xi) \;=\; \prod_{k=1}^{N} F_k(\xi) \,. \tag{H.14}$$

since — according to Eq. (H.9) — the operation of convoluting is commutating and associative. This yields

$$F_{1\circ\ldots\circ N}(z) = \frac{\xi^{f^{\text{tot}}/2}}{(2\pi)^{N/2}} \frac{1}{(\xi - iz)^{f^{\text{tot}}/2}} \,, \quad -\infty < z < \infty \,,\; 0 < \xi < \infty \,, \tag{H.15}$$

where f^{tot} is given by Eq. (H.7).

Inverting the Fourier transformation that gave (H.15), one finds

$$\frac{\xi^{f^{\text{tot}}/2}}{(2\pi)^{(N+1)/2}} \int_{-\infty}^{\infty} \mathrm{d}z \, \frac{e^{-iTz}}{(\xi - iz)^{f^{\text{tot}}/2}} = \begin{cases} \frac{\xi^{f^{\text{tot}}/2}}{\Gamma(f^{\text{tot}}/2)} T^{f^{\text{tot}}/2-1} \exp(-T\xi) & \text{for } T > 0 \\ 0 & \text{for } T < 0 \,, \end{cases} \tag{H.16}$$

see Eqs. (H.33), and (H.26). Thus the quantity T of Eq. (H.6) follows a chi-squared distribution with f^{tot} degrees of freedom which was to be shown.

H.3 Definitions of Fourier Transforms

Let $q(t)$ be a real function which can be integrated over the whole real axis, i.e. the integral

$$\int_{-\infty}^{\infty} \mathrm{d}t \, q(t)$$

exists. We do not require the function $q(t)$ to be regular at all t . The Fourier transform

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}t \, q(t) e^{izt} \tag{H.17}$$

exists for all real z. This definition of is found in Sect. 12.21 of [2].

The Fourier transform is an expansion of q in terms of the orthogonal functions

$$\frac{1}{\sqrt{2\pi}}\, e^{izt} .$$

They are orthogonal in the sense that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, e^{i(z-z')t} \;=\; \delta(z-z') \tag{H.18}$$

according to the entry 1 in Sect. 12.23 of [2].

The inversion of the Fourier transform is given in Sect. 12.21 of [2] to be

$$q(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}z\, F(z)e^{-izt} . \tag{H.19}$$

Antisymmetric and symmetric functions q can be expanded into the so-called Fourier sine and Fourier cosine transforms

$$F_s(z) = \Big(\frac{2}{\pi}\Big)^{1/2}\int_0^{\infty} \mathrm{d}t\, q(t)\sin(zt) \tag{H.20}$$

and

$$F_c(z) = \Big(\frac{2}{\pi}\Big)^{1/2}\int_0^{\infty} \mathrm{d}t\, q(t)\cos(zt)\, , \tag{H.21}$$

see Sect. 12.31 of [2]. The symmetric and antisymmetric parts of $q(t)$ — in the sense of a reflection at the origin — are picked up by the cosine and sine transformations. The symmetric part is

$$q^S(t) = \frac{1}{2}\Big(q(t)+q(-t)\Big) \tag{H.22}$$

while

$$q^A(t) = \frac{1}{2}\Big(f(t)-f(-t)\Big) \tag{H.23}$$

is the antisymmetric part. This leads to

$$\begin{aligned} F(z) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}t\,\Big[q^S(t)+q^A(t)\Big]\Big[\cos(zt)+i\sin(zt)\Big] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}t\,\Big[q^S(t)\cos(zt)+iq^A(t)\sin(zt)\Big] \end{aligned} \tag{H.24}$$

because the integrals over products of a symmetric function with an antisymmetric one vanish. It follows

$$F(z) \;=\; \Big(\frac{2}{\pi}\Big)^{1/2}\int_0^{\infty} \mathrm{d}t\,\Big[q^S(t)\cos(zt)+iq^A(t)\sin(zt)\Big] . \tag{H.25}$$

H.4 The Fourier Transform of the Chi-Squared Distribution

We are interested in the Fourier transform $F(z)$ — defined in (H.17) — of the chi-squared distribution (H.5). This means to derive the Fourier transform $F(z)$ of the distribution

$$q(t|\xi) = \begin{cases} \frac{\xi^{f/2}}{\Gamma(f/2)} t^{f/2-1} \exp(-t\xi) & \text{for } x > 0 \\ 0 & \text{for } x < 0\,. \end{cases} \tag{H.26}$$

For simplicity we have omitted the index k that appears in Eq. (H.5). Thus f is not a function but rather the number of degrees of freedom of the chi-squared distribution. Note that ξ in the present context is a fixed number that conditions the distribution q.

Equation (H.26) defines q such that both, its symmetric and its antisymmetric part, equal $q(t|\xi)/2$ for positive t. In this case Eq. (H.25) yields the Fourier transform

$$F(z) = \frac{1}{2}\Big[F_c(z) + iF_s(z)\Big]\,. \tag{H.27}$$

The Fourier cosine transform F_c is given as entry 14[7] of Sec. 12.34 in [2]. One finds

$$F_c(z) = \Big(\frac{2}{\pi}\Big)^{1/2} \xi^{f/2}(\xi^2 + z^2)^{-f/4} \cos\Big(\frac{f}{2}\tan^{-1}\frac{z}{\xi}\Big)\,. \tag{H.28}$$

The Fourier sine transform is

$$F_s(z) = \Big(\frac{2}{\pi}\Big)^{1/2} \xi^{f/2}(\xi^2 + z^2)^{-f/4} \sin\Big(\frac{f}{2}\tan^{-1}\frac{z}{\xi}\Big)\,, \tag{H.29}$$

see entry 16 of Sec. 12.33 in [2]. Equation (H.27) then gives

$$\begin{aligned} F(z) &= \frac{\xi^{f/2}}{\sqrt{2\pi}}\Big(\xi^2 + z^2\Big)^{-f/4} \exp\Big(i\frac{f}{2}\tan^{-1}\frac{z}{\xi}\Big)\,, \\ -\infty &< z < \infty\,, \\ 0 &< \xi < \infty\,. \end{aligned} \tag{H.30}$$

The arctangent of z/ξ is the phase of the complex number $\xi + iz$, i.e.

$$\tan^{-1}\frac{z}{\xi} = \arg(\xi + iz)\,. \tag{H.31}$$

Therefore Eq. (H.30) takes the form

$$\begin{aligned} F(z) &= \frac{\xi^{f_k/2}}{\sqrt{2\pi}}(\xi^2+z^2)^{-f_k/4}\exp\Big(i\frac{f_k}{2}\arg(\xi+iz)\Big) \\ &= \frac{\xi^{f_k/2}}{\sqrt{2\pi}}\Big[(\xi^2+z^2)^{1/2}\exp\big(-i\arg(\xi+iz)\big)\Big]^{-f_k/2} \\ &= \frac{\xi^{f_k/2}}{\sqrt{2\pi}}(\xi-iz)^{-f_k/2} \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{(1-iz/\xi)^{f_k/2}}\,, \qquad -\infty<z<\infty\,,\ 0<\xi<\infty\,. \end{aligned} \tag{H.32}$$

This is the Fourier transform of the function defined in Eq. (H.26).

The inversion (H.19) of this Fourier transformation yields the function in Eq. (H.26),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}z,\frac{e^{-izt}}{(1-iz/\xi)^{f/2}} = q(t|\xi)\,, \tag{H.33}$$

from which we started.

Bibliography

[1] Ch. Fuhrmann, H.L. Harney, K. Harney, and A. Müller. On the Bayesian posterior distribution. To be published.

[2] I.S. Gradshteyn, I.M. Ryzhik, and D. Zwillinger. *Table of Integrals, Series, and Products*. Academic Press, New York, 8^{th} edition, 2015.