Chapter 1

Knowledge and Logic

Science does not prove anything. Science infers statements about reality. Sometimes the statements are of stunning precision, sometimes they are rather vague. Science never reaches exact results. Mathematics provides proofs but it is devoid of reality. The present book will show in mathematical terms how to express uncertain experience in scientific statements.

Every observation leads to randomly fluctuating results. Therefore the conclusions drawn from them, must be accompanied by an estimate of their truth, usually expressed as a probability. Such a conclusion typically has the form: "The quantity ξ inferred from the present experiment has the value $\alpha \pm \sigma$." An experiment never yields certainty about the true value of ξ. Rather the result is characterised by an interval in which the true value should lie. It does not even lie with certainty in that interval. A more precise interpretation of the above interval is: "The quantity ξ is with the probability $K = 0.68$ in the interval $[\alpha - \sigma, \alpha + \sigma]$." Trying to be even more precise one would say: "We assign a Gaussian distribution to the parameter ξ. The distribution is centered at α and has the standard deviation σ. The shortest interval containing ξ with the probability $K = 0.68$ is then $\alpha \pm \sigma$." In simplified language, the standard deviation of the assumed Gaussian distribution is called "the error" of the result, although "the" error of the result cannot be specified. One is free to choose the probability K and thus the length of the error interval.

The present book generalises the well-known rules of Gaussian error assignments to cases, where the Gaussian model does not apply. Of course, the Gaussian model is treated too. But the book is animated by the question: How to estimate the error interval when the data follow a distribution other than Gaussian, e.g. a Poissonian one? This requires us to answer the following general questions: What is — in any case — the definition of an error interval? How do we understand probability? How should the observed events x be related to the parameter ξ of interest?

1.1 Knowledge

The parameter that one wants to know is never measured directly and immediately. The true length of a stick is hidden behind the random fluctuations of the value that one reads on a meter. The true position of a spectral line is hidden in the line width that one observes with the spectrograph. The fluctuations have different causes in these two cases but they cannot be avoided. One does not observe the interesting parameter ξ. Rather, one observes events x that have a distribution p depending on ξ. Data analysis